A proof of Mansfield's theorem by forcing method

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Under the assumption that, *for every real s, there are only countably many reals constructible from s* (hypothesis P), R. M. Solovay has proved the following theorem [6, Theorem 4]: *Let A be a $\mathbf{\Pi}^1_1$ subset of N^N with code $\alpha \in N^N$, then either every element of A is constructible from α or A contains a perfect set.* By showing that Solovay's original proof can be carried out in a Boolean-valued model for ZF, R. Mansfield obtained an improvement of Solovay's theorem [1, T 3206; 6, Postscript II]: *Let A be $\mathbf{\Sigma}^1_2$, and suppose that A contains a real not constructible from the code for A. Then A contains a perfect set.*

In this paper we shall prove the latter using Cohen's forcing method. The essential points of the proof is to remove the hypothesis P from Solovay's Theorem 4, by carrying out an argument, analogous to that of Solovay, in a suitable countable standard model for ZF without using the hypothesis. As is easily seen, Mansfield's theorem is reduced to the case that A is $\mathbf{\Pi}^1_1$. Hence we shall prove the following:

THEOREM. *Let A be $\mathbf{\Pi}^1_1$, and suppose that A contains a real not constructible from the code for A. Then A contains a perfect set.*

Most of terminologies in this paper are borrowed from [6].

To prove the theorem we must begin with a general lemma.

LEMMA 1. *Let $\mathfrak{M} = \langle M, \in \rangle$ be a transitive model for ZF, then $\mathbf{\Sigma}^1_1$- and $\mathbf{\Pi}^1_1$-predicates are absolute for $\mathfrak{M}$:*

$$(1) \qquad (\exists \xi) R(\alpha, \xi) \equiv (\exists \xi)_M R(\alpha, \xi) ,$$

$$(2) \qquad (\forall \xi) R(\alpha, \xi) \equiv (\forall \xi)_M R(\alpha, \xi) ,$$

where $\alpha \in M$ and R is an arithmetical predicate.

Proof. Since M is transitive, the notion of ordinal, the constant ω, etc. are absolute. Hence so also does an arithmetical predicate. Now let E be the Π^1_1-set defined by

$$\alpha \in E \equiv (\forall \xi) R(\alpha, \xi) ,$$

and $E^{(\mathfrak{M})}$ be its relativization to $\mathfrak{M}$:

$$\alpha \in E^{(\mathfrak{M})} \equiv \alpha \in M \ \& \ (\forall \xi)_M R(\alpha, \xi) .$$

Then

$$E^{(\mathfrak{M})} \supseteq E \cap M = (\bigcup_{\lambda<\omega_1} E_\lambda) \cap M = \bigcup_{\lambda<\omega_1} (E_\lambda \cap M) \supseteq \bigcup_{\lambda<\omega_1^{(\mathfrak{M})}} (E_\lambda \cap M)\ ,$$

where $\bigcup_{\lambda<\omega_1} E_\lambda$ is the canonical decomposition of E into its constituents. Let $E_\lambda^{(\mathfrak{M})}(\lambda<\omega_1^{(\mathfrak{M})})$ be the relativization of E_λ to $\mathfrak{M}$. In relativizing E_λ, chose a code for λ from M. Since the constituent E_λ can be defined by a $\mathbf{\Sigma}_1^1$-predicate,

$$\bigcup_{\lambda<\omega_1^{(\mathfrak{M})}} (E_\lambda \cap M) \supseteq \bigcup_{\lambda<\omega_1^{(\mathfrak{M})}} E_\lambda^{(\mathfrak{M})} = E^{(\mathfrak{M})}\ .$$

Thus we have $E \cap M = E^{(\mathfrak{M})}$. Hence for each $\alpha \in M$,

$$\begin{aligned}(\forall\xi)R(\alpha, \xi) &\equiv \alpha \in E \equiv \alpha \in E \cap M \equiv \alpha \in E^{(\mathfrak{M})} \\ &\equiv \alpha \in M \;\&\; (\forall\xi)_M R(\alpha, \xi) \equiv (\forall\xi)_M R(\alpha, \xi)\ .\end{aligned}$$

This proves (2). (1) follows from (2) by duality.

Let A be a $\mathbf{\Pi}_1^1$-set coded by α (or Π_1^1 in α, in general) and

$$A = \bigcup_{\lambda<\omega_1} A_\lambda$$

be the canonical decomposition of A into its constituents. Suppose that A contains reals not constructible from α. We pick $\lambda_0<\omega_1$ such that A_{λ_0} has reals not constructible from α. Assuming that

(*) $\quad A_{\lambda_0}$ *contains no perfect set* ,

we shall reach a contradiction.

LEMMA 2. *The constituent A_{λ_0} is countable. Any element of A_{λ_0} is hyperarithmetical in α and a code for λ_0. A_{λ_0} has no real constructible from α.*[1]

Proof. If A_{λ_0} is uncountable, then there must be a perfect set P such that

$$P \subseteq A_{\lambda_0} \subseteq A\ ,$$

a contradiction to the assumption (*). The same is the case when A_{λ_0} contains a real not hyperarithmetical in α and a code for λ_0 [1, T 3200]. Suppose that A_{λ_0} contains a real constructible from α, then there is a code for λ_0 constructible from α. Since all elements of A_{λ_0} are hyperarithmetical in α and this code, by Lemma 1, they must be constructible from α.

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1) The last assertion of this lemma, which, evident as it is, makes the whole proof much simpler, is suggested by Professor H. Tanaka.

Let $\beta_0 \in A_{\lambda_0}$ be fixed. Hereafter we must confine our consideration to $L[\alpha, \beta_0]$. Practically, however, no trouble can arise if we confuse $L[\alpha, \beta_0]$ with the real world, because we deal exclusively with $\boldsymbol{\Sigma}_2^1$- and $\boldsymbol{\Pi}_2^1$-predicates which are absolute for $L[\alpha, \beta_0]$ (cf. [5]). It should be noticed that A_{λ_0} is absolute for $L[\alpha, \beta_0]$ and that the countability of A_{λ_0} remains valid still in $L[\alpha, \beta_0]$ (cf. [3]). Designate by ZFL' the theory obtained from ZF by adding a new axiom $V = L[\alpha, \beta_0]$.

LEMMA 3. *There is a conservative extension ZFL'_{A} of ZFL' having an element M such that $\mathfrak{M} = \langle M, \in \rangle$ is a countable transitive model for ZF in ZFL'_{A} and that $\alpha \in M$, $\lambda_0 \in M$ and $M \cap A_{\lambda_0} = \varnothing$.*[1]

Proof. In the lemma in [4, p. 279], replace ZFL by ZFL' and let B be the transitive closure of $\{\alpha, \beta_0\}$. Then the system $\langle \mathrm{A}, \in \rangle$ is a countable transitive model for ZF, $\alpha \in \mathrm{A}$ and $\lambda_0 \in \mathrm{A}$. Let M be the collection of all elements constructible from α in A as the universe. Then $\mathfrak{M} = \langle M, \in \rangle$ is also countable transitive model for ZF, $\alpha \in M$ and $\lambda_0 \in M$. Moreover $M \cap A_{\lambda_0} = \varnothing$, since an ordinal in A is a real ordinal and no element of A_{λ_0} is constructible from α.

Now, we shall make use of Cohen's forcing method to construct a code for λ_0. Our ground model is the above $\mathfrak{M}$. Let $\boldsymbol{\gamma}$ be a function symbol, which is intended to denote a code for λ_0. We construct as the usual way the ramified language $\mathscr{L}[\boldsymbol{\gamma}]$. A *forcing condition* is a partial injective function

$$p : \omega \longrightarrow \lambda_0$$

with a finite domain. They are designated by $p, q, p_0, p_1, \cdots$. A condition p *forces* $\boldsymbol{\gamma}(u) = v$, in symbol $p \Vdash \boldsymbol{\gamma}(u) = v$, iff

$$(\exists m \in \omega)(\exists n \in \omega)[p \Vdash u = 2^m 3^n \ \& \ \{(p \Vdash v = 0 \ \& \ p(m) < p(n)) \vee (p \Vdash v = 1 \\ \& \ p(n) \leq p(m))\}] \vee (\forall m \in \omega)(\forall n \in \omega)[p \Vdash \sim (u = 2^m 3^n) \ \& \ p \Vdash v = 1] .$$

The remaining articles of the definition of forcing are the same as usual. The complete sequence is the original one of Conen: an increasing sequence of conditions is *complete* iff each sentence of $\mathscr{L}[\boldsymbol{\gamma}]$ is eventually forced by conditions of the sequence.

In the real world let γ be a code for λ_0. There is a predicate $\Phi(\beta, \alpha, \gamma)$, Π_1^1 in α and γ, which defines an element $\beta \in A_{\lambda_0}$ (Cf. [2]): $(\exists !\, \xi)\Phi(\xi, \alpha, \gamma)$ and $(\forall \xi)[\Phi(\xi, \alpha, \gamma) \rightarrow \xi \in A_{\lambda_0}]$. Let t be designate the following abstraction term in $\mathscr{L}[\boldsymbol{\gamma}]$:

$$\hat{m}\hat{n}(\exists \xi)[\Phi(\xi, \boldsymbol{\alpha}, \boldsymbol{\gamma}) \ \& \ \langle m, n \rangle \in \xi] .$$

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[1] Terminologies and notations in this lemma and its proof borrowed from [4]. The roman capital A used here stand for the A in the lemma in [4, p. 279].

LEMMA 4. *For any choice of generic γ, the denotation $\bar{t}$ of t belongs to A_{λ_0}. Hence $\bar{t}$ is nonconstructible from α.*

Proof. Let $\mathfrak{N}=\langle M[\gamma], \in\rangle$ be the Cohen extension of $\mathfrak{M}$ for the γ. Then $A_{\lambda_0}\subseteq M[\gamma]$, since an element of A_{λ_0} is hyperarithmetical in γ. Let β be a real such that $\Phi(\beta, \alpha, \gamma)$, then $\beta\in A_{\lambda_0}\subseteq M[\gamma]$. Hence by Lemma 1, $\Phi^{(\mathfrak{N})}(\beta, \alpha, \gamma)$. Therefore $(\exists\xi)_{M[\gamma]}\Phi^{(\mathfrak{N})}(\xi, \alpha, \gamma)$. Again by Lemma 1,

$$\begin{aligned}
&(\forall\xi)(\forall\eta)[\Phi(\xi, \alpha, \gamma) \,\&\, \Phi(\eta, \alpha, \gamma) \rightarrow \xi = \eta]\\
\supset\ &(\forall\xi)_{M[\gamma]}(\forall\eta)_{M[\gamma]}[\Phi(\xi, \alpha, \gamma) \,\&\, \Phi(\eta, \alpha, \gamma) \rightarrow \xi = \eta]\\
\equiv\ &(\forall\xi)_{M[\gamma]}(\forall\eta)_{M[\gamma]}[\Phi^{(\mathfrak{N})}(\xi, \alpha, \gamma) \,\&\, \Phi^{(\mathfrak{N})}(\eta, \alpha, \gamma) \rightarrow \xi = \eta]\,.
\end{aligned}$$

Thus $(\exists!\,\xi)\Phi(\xi, \alpha, \gamma)$ holds in $\mathfrak{N}$. And so also does $(\forall\xi)[\Phi(\xi, \alpha, \gamma)\rightarrow\xi\in A_{\lambda_0}]$. Let $\beta_0\in A_{\lambda_0}$ be the unique element that satisfies $\Phi^{(\mathfrak{N})}(\beta_0, \alpha, \gamma)$. Then

$$\begin{aligned}
\langle m, n\rangle\in\bar{t} &\equiv \models_{\mathfrak{N}}(\exists\xi)[\Phi(\xi, \boldsymbol{\alpha}, \boldsymbol{\gamma}) \,\&\, \langle \boldsymbol{m}, \boldsymbol{n}\rangle\in\xi]\\
&\equiv (\exists\xi)_{M[\gamma]}[\Phi^{(\mathfrak{N})}(\xi, \alpha, \gamma) \,\&\, \langle m, n\rangle\in\xi]\\
&\equiv (\exists\xi)_{M[\gamma]}[\xi = \beta_0 \,\&\, \Phi^{(\mathfrak{N})}(\xi, \alpha, \gamma) \,\&\, \langle m, n\rangle\in\xi]\\
&\quad\ \vee(\exists\xi)_{M[\gamma]}[\xi \neq \beta_0 \,\&\, \Phi^{(\mathfrak{N})}(\xi, \alpha, \gamma) \,\&\, \langle m, n\rangle\in\xi]\\
&\equiv \Phi^{(\mathfrak{N})}(\beta_0, \alpha, \gamma) \,\&\, \langle m, n\rangle\in\beta_0\\
&\equiv \langle m, n\rangle\in\beta_0\,.
\end{aligned}$$

Hence $\bar{t}=\beta_0$. Therefore $\bar{t}\in A_{\lambda_0}$.

Let $\beta\in N^N$. Following Solovay, we say that a condition p *makes* t *unequal to* β iff there are $m, n\in\omega$ such that $\beta(m)=n$ and $p\Vdash\sim\langle\boldsymbol{m}, \boldsymbol{n}\rangle\in t$.

LEMMA 5. *For any real β and any condition p, there is a $q\supseteq p$ which makes t unequal to β.*

Proof. Let $T=\{\langle m, n\rangle\in\omega\times\omega : p\Vdash^{*}\langle\boldsymbol{m}, \boldsymbol{n}\rangle\in t\}$, where $\Vdash^{*}$ designates "weak forcing". The remainder of the proof is the same as in [6].

Obviously,

LEMMA 6. *Suppose that p makes t unequal to β and let $\{p_n\}$ be a complete sequence such that $p\subseteq q_n$ for same n. Then the denotation $\bar{t}$ of t induced by $\{p_n\}$ is not equal to β.*

Proof of Theorem. Let $\beta_0, \beta_1, \cdots, \beta_n, \cdots$ be an enumeration of the elements of A_{λ_0}, and $\Phi_0, \Phi_1, \cdots, \Phi_n, \cdots$ an enumeration of all sentences in $\mathscr{L}[\boldsymbol{\gamma}]$. We construct a complete sequence $\{p_n\}$ as follows: $p_0=\varnothing$. Suppose that $p_0\subseteq p_1\subseteq\cdots\subseteq p_n$ are already choosen. There is a $q_{n+1}\supseteq p_n$ such that either $q_{n+1}\Vdash\Phi_n$ or $q_{n+1}\Vdash\sim\Phi_n$. By Lemma 5, there is a $p_{n+1}\supseteq q_{n+1}$ which makes t unequal to β_n.

Let $\bar{t}$ be the denotation of t induced by $\{p_n\}$. Then, by Lemma 4,

$\bar{t} \in A_{\lambda_0}$. On the other hand, by the definition of the complete sequence $\{p_n\}$ and by Lemma 6, $\bar{t} \neq \beta_n$ $(n = 0, 1, \cdots)$. Thus a contradiction is reached in ZFL'_A. So also does in ZF, since ZFL'_A is a conservative extension of ZFL' and Consis $(ZF) \rightarrow$ Consis (ZFL'). This disproves $(*)$.

The CONVERSE OF THE THEOREM also holds: *Let A be a* $\boldsymbol{\Pi}^1_1$*-set coded by* α*. Suppose that the real world has reals nonconstructible from* α*. If A contains a perfect set, then A has reals nonconstructible from* α*.*

Proof. When A contains a perfect set, then A contains Cantor's discontinuum. As is well known in elementary analysis, there is a one to one bicontinuous function from N^N into Cantor's discontinuum. Hence applying the method of effective choice [3], we obtain a function from N^N into A which is defined by a Δ^1_2-predicate in α. The image of a real nonconstructible from α by this function is nonconstructible from α.

Finally we can easily prove Solovay's original result:

COROLLARY. *Every uncountable* $\boldsymbol{\Sigma}^1_2$*-set of reals contains a perfect set iff the reals constructible from a real are always countable.*

References

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Added in Proof. The auther found that Lemma 1 is Mostowski's result. Cf. [1, T 3100].

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